

A remark on conditions that a diffusion in the natural scale is a martingale

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January 24, 2017

Abstract

We consider a diffusion processes $\{X_t\}$ on an interval in the natural scale. Some results are known under which $\{X_t\}$ is a martingale, and we give simple and analytic proofs for them.

Mathematics Subject Classification (2010): 60J60, 60G44

1 Introduction

Let $-\infty \leq l_- < l_+ \leq \infty$ and let m be a Borel measure with $\text{supp } m = (l_-, l_+)$. We denote by $\{\{X_t\}_{t \geq 0}, \{P_x\}_{x \in (l_-, l_+)}\}$ the minimal diffusion process on (l_-, l_+) with the speed measure m and the scale function $s(x) = x$. It is well known that a local martingale $\{X_t\}$ is a martingale if and only if $\{X_T : T \text{ is a stopping time with } T \leq t\}$ is uniformly integrable for any $t \geq 0$. Here our aim is to have more explicit condition for the one-dimensional diffusions in the natural scale. If $|l_{\pm}| < \infty$, $\{X_t\}$ is bounded so that it is a martingale. If $l_- = -\infty$, $l_+ < \infty$, this can be reduced to the case of $l_- < \infty$, $l_+ = \infty$ by replacing X_t by $-X_t$. Hence it suffices to consider the following two cases.

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Case I : $-\infty < l_-, l_+ = +\infty$, Case II : $l_- = -\infty, l_+ = +\infty$.

Let $P(l_-, l_+)$ be the set of Borel measures on (l_-, l_+) , and for $\mu \in P(l_-, l_+)$ let $P_\mu(\cdot) := \int_{(l_-, l_+)} P_x(\cdot) \mu(dx)$. According to Lemma 4.1 ([1], Lemma 2), $\{X_t^\tau\}$ is a P_μ -martingale for some $\mu \in P(l_-, l_+)$ with $\int_{(l_-, l_+)} |x| \mu(dx) < \infty$ if and only if $\{X_t^\tau\}$ is P_x -martingale for any $x \in (l_-, l_+)$. We further set

$$\tau_a := \inf \{t \geq 0 | X_t = a\}, \quad \tau_\pm := \lim_{a \rightarrow l_\pm} \tau_a, \quad \tau := \tau_+ \wedge \tau_-$$

$$X_t^\tau := X_{t \wedge \tau}.$$

Kotani [1] showed the following theorem.

Theorem 1.1 [1] *$\{X_t^\tau\}$ is a P_x -martingale for any $x \in (l_-, l_+)$ if and only if*

Case I :

$$\int_{[r, l_+)} xm(dx) = \infty, \quad r \in (l_-, \infty)$$

Case II :

$$\int_{[r, l_+)} xm(dx) = \infty \quad \text{and} \quad \int_{(l_-, r]} |x| m(dx) = \infty, \quad r \in (-\infty, \infty).$$

By Feller's criterion, $P_x(\tau_\# = \infty) = 1$ if $|l_\#| = \infty$, $\# = \pm\infty$. Thus Theorem 1.1 implies that $\{X_t^\tau\}$ is a martingale if and only if the boundaries at infinity are natural. Hulley, Platen [2] derived another condition. Let

$$\mathcal{L}f := \frac{d^2}{dm dx} f$$

be the generator of $\{X_t\}$ and for $\lambda > 0$ let f_- (resp. f_+) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L}f = \lambda f$, which are unique up to constants unless the boundary is regular.

Theorem 1.2 [2] *$\{X_t^\tau\}$ is a P_x -martingale for any $x \in (l_-, l_+)$ if and only if*

Case I :

$$\lim_{z \rightarrow \infty} f'_-(z) = \infty$$

Case II :

$$\lim_{z \rightarrow \infty} f'_-(z) = \infty \quad \text{and} \quad \lim_{z \rightarrow -\infty} f'_+(z) = -\infty.$$

Gushchin, Urusov, and Zervos [3] derived a condition that $\{X_t^\tau\}$ is a submartingale or a supermartingale.

Theorem 1.3 [3] *$\{X_t^\tau\}$ is a P_x -submartingale if and only if $\int_r^\infty xm(dx) = \infty$, $r \in (l_-, l_+)$.*

By [2] Proposition 3.16, 3.17, this condition is equivalent to $\lim_{t \rightarrow \infty} f'_-(t) = \infty$. Together with Theorem 1.3 we thus have

Theorem 1.4 *$\{X_t^\tau\}$ is a P_x -submartingale if and only if $\lim_{t \rightarrow \infty} f'_-(t) = \infty$.*

Moreover in [3], they further derived a condition in Case I such that $\{X_t^\tau\}$ is a strict P_x supermartingale, that is, $\{X_t^\tau\}$ is a P_x -supermartingale but is not a P_x -martingale.

Theorem 1.5 [3] *Let $-\infty < l_-$, $l_+ = \infty$. Then $\{X_t^{\tau-}\}$ is a strict P_x -supermartingale if and only if*

$$\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau-}] = l_-$$

for any $x \in (l_-, l_+)$.

We believe that Theorem 1.5 is also true for $l_- = -\infty$. The goal of this paper is :

- (1) To give a simple analytic proof of Theorem 1.4 without using the results in [2]. We note that the proofs of Proposition 3.16, 3.17 in [2] is more or less probabilistic using Tanaka's formula.
- (2) To give a simple analytic proof of Theorem 1.5 ; the original proof of that in [3] is done by embedding $\{X_t\}$ into the geometric Brownian motion on the torus.

The rest of this paper is organized as follows. In Section 2 (resp. Section 3), we give a proof of Theorem 1.4 (resp. Theorem 1.5). In Appendix, we prepare some tools for these proofs.

2 A proof of Theorem 1.4

In Case I, the statement follows from Theorem 1.2, for $\{X_t^{\tau-}\}$ is always a P_x -supermartingale being bounded from below. Henceforth we consider Case II.

Suppose $\{X_t\}$ is a P_x -submartingale and let $z < x$. Then $\{X_t^{\tau_z}\}$ is bounded from below so that it is a P_x -martingale. For $\lambda > 0$, let f_-^z (resp. f_+^z) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L}f = \lambda f$ such that $f_-^z(z) = 0$. Then we have

$$f_-^z(x) = f_-(x) - \frac{f_-(z)}{f_+(z)} f_+(x), \quad f_+^z(x) = f_+(x).$$

Since f'_+ is increasing, we have

$$f'_-(x) = f_-^{z'}(x) + \frac{f_-(z)}{f_+(z)} f'_+(x) \geq f_-^{z'}(x) + \frac{f_-(z)}{f_+(z)} f'_+(z), \quad x \in (z, \infty).$$

Applying Theorem 1.2 to $\{X_t^{\tau_z}\}$ yields $\lim_{t \rightarrow \infty} f_-^{z'}(t) = \infty$ and thus $\lim_{t \rightarrow \infty} f'_-(t) = \infty$.

Conversely, suppose $\lim_{t \rightarrow \infty} f'_-(t) = \infty$ and let $z < x$. Then

$$\begin{aligned} \lim_{z \rightarrow \infty} z \int_0^\infty e^{-\lambda t} P_x(\tau_z < t) dt &= \lim_{z \rightarrow \infty} \frac{z}{\lambda} E_x[e^{-\lambda \tau_z}] = \lim_{z \rightarrow \infty} \frac{z}{\lambda} \frac{f_-(x)}{f_-(z)} \\ &= \lim_{z \rightarrow \infty} \frac{f_-(x)}{\lambda} \frac{1}{f'_-(z)} = 0 \end{aligned}$$

where we used Lemma 4.3 and l'Hospital's rule. By Fatou's lemma,

$$\int_0^\infty e^{-\lambda t} \liminf_{z \rightarrow \infty} z P_x(\tau_z < t) dt = 0.$$

Hence $\liminf_{z \rightarrow \infty} z P_x(\tau_z < t) = 0$ so that we can find a sequence $\{z_n\} \subset (x, \infty)$ with $\lim_{n \rightarrow \infty} z_n = \infty$ such that

$$\lim_{n \rightarrow \infty} z_n P_x(\tau_{z_n} < t) = 0.$$

On the other hand $\{X_t^{\tau_{z_n}}\}$ is a P_x -submartingale being bounded from above and

$$x \leq E_x[X_{t \wedge \tau_{z_n}}] = z_n P_x(\tau_{z_n} < t) + E_x[X_t; \tau_{z_n} \geq t].$$

Since $\lim_{n \rightarrow \infty} P_x(\tau_{z_n} \geq t) = 1$, $x \leq E_x[X_t]$. Markov property implies $\{X_t\}$ is a P_x -submartingale. \square

3 A proof of Theorem 1.5

Without losing generality, we may suppose $l_- < 0$. For $\lambda > 0$, let f_- (resp. f_+) be the positive increasing (resp. positive decreasing) solution to the equation $\mathcal{L}f = \lambda f$ such that $f_-(l_-) = 0$. Let G be Green's function of \mathcal{L} :

$$G(x, y, \lambda) := \begin{cases} \frac{1}{h} f_-(y) f_+(x) & (y < x) \\ \frac{1}{h} f_-(x) f_+(y) & (x \leq y) \end{cases}$$

$$h := f_+(x) f'_-(x) - f_-(x) f'_+(x).$$

Then we have

$$\int_{l_-}^{\infty} G(x, y, \lambda) (y - l_-) m(dy) = E_x \left[\int_0^{\infty} e^{-\lambda t} (X_{t \wedge \tau_-} - l_-) dt \right]. \quad (3.1)$$

Let $\alpha_+ := \lim_{t \rightarrow \infty} f_+(t)$. Then $f'_+ \in L^1(a, \infty)$ for $a \in (l_-, \infty)$ and

$$f_+(x) = \alpha_+ - \int_x^{\infty} f'_+(y) dy.$$

Therefore $\lim_{x \rightarrow \infty} f'_+(x) = 0$. The equation $\mathcal{L}f_+ = \lambda f_+$ yields

$$f'_+(x) = -\lambda \int_x^{\infty} f_+(y) m(dy)$$

$$f_+(x) = \alpha_+ + \lambda \int_x^{\infty} (y - x) f_+(y) m(dy)$$

so that we have

$$\lambda \int_x^{\infty} y f_+(y) m(dy) = f_+(x) - \alpha_+ - x f'_+(x).$$

Similarly,

$$f'_-(y) = f'_-(l_-) + \lambda \int_{l_-}^y f_-(z) m(dz)$$

$$f_-(x) = f'_-(l_-)(x - l_-) + \lambda \int_{l_-}^x (x - y) f_-(y) m(dy)$$

$$\lambda \int_{l_-}^x y f_-(y) m(dy) = f'_-(l_-)(x - l_-) - f_-(x) + \lambda x \int_{l_-}^x f_-(y) m(dy).$$

Substituting them into (3.1) yields

$$\int_0^{\infty} e^{-\lambda t} E_x [X_{t \wedge \tau_-} - l_-] dt = \frac{x - l_-}{\lambda} - \frac{\alpha_+ f_-(x)}{\lambda h}. \quad (3.2)$$

We note that (3.2) and Lemma 4.1 also proves Theorem 1.1 in Case I.

Suppose $\{X_t^{\tau-}\}$ is a strict P_x -supermartingale. The discussion above implies $\alpha_+ > 0$. We shall show below that

$$\lim_{\lambda \rightarrow 0} \left(x - l_- - \frac{\alpha_+ f_-(x)}{h} \right) = 0. \quad (3.3)$$

Let ϕ, ψ be the solution to $\mathcal{L}f = \lambda f$ with the initial condition

$$\begin{aligned} \phi(0) &= 1, & \phi'(0) &= 0 \\ \psi(0) &= 0, & \psi'(0) &= 1. \end{aligned}$$

Then f_{\pm} satisfy

$$f_+(x) = \phi(x) - \left(\lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} \right) \psi(x), \quad f_-(x) = \phi(x) - \left(\lim_{x \rightarrow l_-} \frac{\phi(x)}{\psi(x)} \right) \psi(x).$$

ψ, ϕ can be composed by the method of successive approximation :

$$\begin{aligned} \phi(x) &= 1 + \sum_{n=1}^{\infty} \lambda^n \phi_n(x), & \phi_0(x) &= 1, & \phi_n(x) &= \int_0^x (x-y) \phi_{n-1}(y) m(dy) \\ \psi(x) &= x + \sum_{n=1}^{\infty} \lambda^n \psi_n(x), & \psi_0(x) &= x, & \psi_n(x) &= \int_0^x (x-y) \psi_{n-1}(x) m(dy) \end{aligned}$$

which is convergent locally uniformly w.r.t. λ [4] which yields

$$\lim_{\lambda \rightarrow 0} \phi(x) = 1, \quad \lim_{\lambda \rightarrow 0} \phi'(x) = 0, \quad \lim_{\lambda \rightarrow 0} \psi(x) = x, \quad \lim_{\lambda \rightarrow 0} \psi'(x) = 1.$$

Moreover

$$\lim_{\lambda \rightarrow 0} \left(- \lim_{x \rightarrow l_-} \frac{\psi(x)}{\phi(x)} \right) = \lim_{\lambda \rightarrow 0} \left(\int_{l_-}^0 \frac{1}{(\phi(x))^2} dx \right) = \int_{l_-}^0 dx = -l_-$$

implies

$$\lim_{\lambda \rightarrow 0} f_-(x) = 1 - \frac{x}{l_-}, \quad \lim_{\lambda \rightarrow 0} f'_-(x) = -\frac{1}{l_-}.$$

On the other hand, by $\alpha_+ > 0$ and by Lemma 4.2, we have $\int_r^{\infty} x m(dx) < \infty$, $r \in (l_-, \infty)$ so that we can find g satisfying

$$g(x) = 1 + \lambda \int_x^{\infty} (y-x) g(y) m(dy)$$

by successive approximation. Using $\alpha_+ > 0$, $\lim_{t \rightarrow \infty} f'_+(t) = 0$, $\lim_{t \rightarrow \infty} g(t) = 1$ and $\lim_{t \rightarrow \infty} g'(t) = 0$, we have

$$f_+(x)g'(x) - f'_+(x)g(x) = 0$$

which implies $f_+(x) = Cg(x)$ for some positive constant C . Because $\lim_{\lambda \rightarrow 0} g(x) = 1$, $\lim_{\lambda \rightarrow 0} g'(x) = 0$,

$$\lim_{\lambda \rightarrow 0} f_+(x) = C, \quad \lim_{\lambda \rightarrow 0} f'_+(x) = 0.$$

Therefore

$$\lim_{\lambda \rightarrow 0} \left(x - l_- - \frac{\alpha_+ f_-(x)}{h} \right) = x - l_- - \frac{C \left(1 - \frac{x}{l_-} \right)}{C \cdot \left(\frac{-1}{l_-} \right) - 0 \cdot \left(1 - \frac{x}{l_-} \right)} = 0$$

proving (3.3). Since $X_{t \wedge \tau_-}$ is a supermartingale, $f(t) := E_x[X_{t \wedge \tau_-} - l_-] \in C^1[0, \infty)$ is monotone decreasing which shows that $\lim_{t \rightarrow \infty} f(t)$ exists and $f' \in L^1(0, \infty)$. Thus by (3.2) and Lemma 4.4

$$\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau_-} - l_-] = 0.$$

Conversely, suppose that $\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau_-} - l_-] = 0$. Then

$$\lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} E_x[X_{t \wedge \tau_-} - l_-] dt = 0$$

which implies $\alpha_+ > 0$ since otherwise it would contradict to (3.2), (3.3). Therefore $\{X_t^{\tau_-}\}$ is not a martingale. \square

4 Appendix

Lemma 4.1 (Lemma 2 in [1])

Suppose $\{X_{t \wedge \tau_-}\}$ is a P_μ -martingale for some $\mu \in P(l_-, \infty)$. Then for any $t \geq 0$, $x \in (l_-, \infty)$,

$$E_x[X_{t \wedge \tau_-}] = x. \tag{4.1}$$

Conversely, if (4.1) is valid, then $\{X_{t \wedge \tau_-}\}$ is a P_μ -martingale for any $\mu \in P(l_-, \infty)$ with $\int_{l_-}^\infty |x| \mu(dx) < \infty$.

Lemma 4.2 *Let $\lambda > 0$ and let f_+ be the positive decreasing solution to $\mathcal{L}f = \lambda f$ with $\alpha_+ := \lim_{x \rightarrow \infty} f_+(x)$. Then the following three conditions are equivalent.*

- (1) $\alpha_+ = 0$
- (2) $\int_a^\infty y m(dy) = \infty$
- (3) $\lambda \int_x^\infty (y - x) f_+(y) m(dy) = f_+(x)$.

Lemma 4.3 *Let f_\pm be the ones defined in the proof of Theorem 1.5. Then*

$$E_x[e^{-\lambda \tau_a}] = \frac{f_+(x)}{f_+(a)}, \quad a < x$$

$$E_x[e^{-\lambda \tau_b} : \tau_b < \tau_-] = \frac{f_-(x)}{f_-(b)}, \quad -\infty \leq l_- < x < b.$$

Lemma 4.4 *Suppose $f \in C^1[0, \infty)$ and $f' \in L^1(0, \infty)$. Then*

- (1) $\lim_{t \rightarrow \infty} f(t)$ exists, and
- (2) $\lim_{t \rightarrow \infty} f(t) = \lambda \lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} f(t) dt$.

References

- [1] S. Kotani, : On a condition that one-dimensional diffusion processes are martingales, In memoriam Paul-Andr e Meyer: S eminaire de Probabilit es XXXIX, Lecture Notes in Mathematics, **1874**(2006), pp.149 - 156, Springer.
- [2] H. Hulley, E. Platen, : A visual classification of local martingales, Quantitative Finance Research Centre, Research Paper **238**(2008), University of Technology, Sydney
- [3] A. Gushchin, M. Urusov, and M. Zervos, : On the submartingale / supermartingale property of diffusions in natural scale, Proceedings of the Steklov Institute of Mathematics Vol. **287**(2014), pp 122-132.
- [4] K. Ito, : Essentials of stochastic process, Translations of mathematical monographs, vol. 231, AMS.